Non shape regular domain decompositions: an analysis using a stable decomposition in H_0^1

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Summary. In this paper, we establish the existence of a stable decomposition in the Sobolev space H_0^1 for domain decompositions which are not shape regular in the usual sense. In particular, we consider domain decompositions where the largest subdomain is significantly larger than the smallest subdomain. We provide an explicit upper bound for the stable decomposition that is independent of the ratio between the diameter of the largest and the smallest subdomain.

1 Introduction

One of the great success stories in domain decomposition methods is the invention and analysis of the additive Schwarz method by Dryja and Widlund in [2]. Even before the series of international conferences on domain decomposition methods started, Dryja and Widlund presented a variant of the historical alternating Schwarz method invented by Schwarz in [5] to prove the Dirichlet principle on general domains. This variant, called the additive Schwarz method, has the advantage of being symmetric for symmetric problems, and it also contains a coarse space component. In a fully discrete analysis in [2], Dryja and Widlund proved, based on a stable decomposition result for shape regular decompositions, that the condition number of the preconditioned operator with a decomposition into many subdomains only grows linearly as a function of $\frac{H}{\delta}$, where H is the subdomain diameter, and δ is the overlap between subdomains. This analysis inspired a generation of numerical analysts, who used these techniques in order to analyze many other domain decomposition methods, see the reference books [6, 4, 7], or the monographs [8, 1], and references therein.

The key assumption that the decomposition is shape regular is, however, often not satisfied in practice: because of load balancing, highly refined subdomains are often physically much smaller than subdomains containing less refined elements, and it is therefore of interest to consider domain decompositions that are only locally shape regular, *i.e.*, domain decompositions where the largest subdomain can be considerably larger than the smallest subdomain, and therefore the subdomain diameter and overlap parameters depend strongly on the subdomain index. In such a domain decomposition, the generic ratio $\frac{H}{\delta}$ from the classical convergence result of the additive Schwarz method can be given at least two different meanings: let H_i refer to the diameter of subdomain number i and δ_i refer to the width of the overlap around subdomain number i. Then in the classical convergence result from [2], one could replace the generic ratio $\frac{H}{\delta}$ by $\frac{\max_i(H_i)}{\min_i(\delta_i)}$, but this is likely to lead to a very pessimistic estimate for the condition number growth. The general analysis of the additive Schwarz method based on a shape regular decomposition does unfortunately not permit to answer the question if the condition number growth for a locally shape regular decomposition is in fact only linear in the quantity $\max_i(\frac{H_i}{\delta_i})$, which is much smaller than $\frac{\max_i(H_i)}{\min_i(\delta_i)}$ in the case of subdomains and overlaps of widely different sizes, a case of great interest in applications.

In [3], we established the existence of a stable decomposition in the continuous setting with an explicit upper bound and a quantitative definition of shape regularity in two spatial dimensions. The explicit upper bound is also linear in the generic quantity $\frac{H}{\delta}$, and the result is limited to shape regular domain decompositions where all subdomains have similar size and where the overlap width is uniform over all subdomains. Having explicit upper bounds, however, allows us now, using similar techniques, to establish the existence of a stable decomposition in the continuous setting with explicit upper bounds when $\max_i(H_i) \gg \min_i(H_i)$, and we provide an explicit upper bound which is linear in $\max_i(H_i/\delta_i)$ for problems in two spatial dimensions. To get this result, only a few of the inequalities established in [3] need to be reworked, and it would be very difficult to obtain such a result without the explicit upper bounds from the continuous analysis in [3].

We state first in §2 our main theorem along with the assumptions we make on the domain decomposition. We then prove the main theorem in §3 in two steps: first, we show in Lemma 1 how to construct the fine component in §3.1, which is an extension of the result [3, Theorem 4.6] for the case where subdomain sizes H_i and overlaps δ_i can strongly depend on the subdomain index i. The major contribution is however in the second step, presented in Lemma 2 in §3.2, where we show how to construct the coarse component in the case of strongly varying H_i and δ_i between subdomains. This result is a substantial generalization of [3, Lemma 5.7]. Using these two new results, and the remaining estimates from [3] which are still valid, we can prove our main theorem. We finally summarize our results in the conclusions in §4.

2 Geometric parameters and main theorem

In the remainder of this paper, we always consider a domain decomposition that has the following properties:

- Ω is a bounded domain of $\mathbb{R}^2$.

- The $(U_i)_{1\leq i\leq N}$ are a non-overlapping domain decomposition of Ω, *i.e.*, satisfy $\bigcup_{i=1}^N \overline{U}_i = \overline{\Omega}$ and $U_i \cap U_j = \emptyset$ when $i \neq j$. The U_i are bounded connected open sets of $\mathbb{R}^2$ and for all subdomains U_i the measure of $\overline{U}_i \setminus U_i$ is zero.
- We set $H_i := \mathrm{diam}(U_i)$.
- Two distinct subdomains U_i and U_j are said to be neighbors if $\overline{U}_i \cap \overline{U}_j \neq \emptyset$.
- For each subdomain U_i, let $\delta_i > 0$ be such that $2\delta_i \leq \min_{j,\overline{U}_i\cap\overline{U}_j=\emptyset}(\mathrm{dist}(U_i,U_j))$. We set $\Omega_i := \{\boldsymbol{x} \in \Omega,\ \mathrm{dist}(\boldsymbol{x},U_i) < \delta_i\}$. The Ω_i form an overlapping domain decomposition of Ω. When subdomains U_i and U_j are neighbors, then the overlap between Ω_i and Ω_j is $\delta_i + \delta_j$ wide. The intersection $\Omega_i \cap \Omega_j$ is empty if and only if the distance between U_i and U_j is positive.
- We set $\delta_i^s = \min_{j\neq i,\overline{U}_i\cap\overline{U}_j\neq\emptyset} \delta_j$ and $\delta_i^l = \max_{j\neq i,\overline{U}_i\cap\overline{U}_j\neq\emptyset} \delta_j$.
- The domain decomposition has N_c colors: there exists a partition of $\mathbb{N}\cap[1,N]$ into N_c sets I_k such that $\Omega_i \cap \Omega_j$ is empty whenever $i \neq j$ and i and j belong to the same color I_k.
- $\mathscr{T}$ is a coarse triangular mesh of Ω: one node $\boldsymbol{x}_i$ per subdomain Ω_i (not counting the nodes located on $\partial\Omega$). By $P_1(\mathscr{T})$, we denote the standard finite element space of continuous functions that are piecewise linear over each triangular cell of $\mathscr{T}$.
- Let θ_{min} be the minimum of all angles of mesh $\mathscr{T}$.
- No node (including the nodes located on $\partial\Omega$) of the coarse mesh has more than K neighbors.
- Let d_i be the length of the largest edge originating from node $\boldsymbol{x}_i$ in the mesh $\mathscr{T}$.
- Let $H_{h,i}$ be the length of the shortest height through $\boldsymbol{x}_i$ of any triangle in the coarse mesh $\mathscr{T}$ that connects to $\boldsymbol{x}_i$. We also set $H'_{h,i}$ as the minimum of $H_{h,j}$ over i and its direct neighbors in mesh $\mathscr{T}$.
- We suppose that for each subdomain U_i, there exists $r_i > 0$ such that U_i is star-shaped with respect to any point in the ball $B(\boldsymbol{x}_i, r_i)$. We also suppose $r_i \leq \frac{H_{h,i}}{4K+1}$ and $r_i \leq H'_{h,i}/2$.
- We also assume the existence of both a pseudo normal $\boldsymbol{X}_i$ and of a pseudo curvature radius $\tilde{R}_i$ for the domain U_i, *i.e.*, we suppose that for each U_i there exists an open layer L_i containing ∂U_i, a vector field $\boldsymbol{X}_i$ continuous on $L_i \cap \overline{U}_i$, $\mathscr{C}^\infty$ on $L_i \cap U_i$ such that $\mathrm{D}\boldsymbol{X}_i(\boldsymbol{x})(\boldsymbol{X}_i(\boldsymbol{x})) = 0$, $\|\boldsymbol{X}_i(\boldsymbol{x})\| = 1$, and $\varepsilon_0 > 0$ such that for all positive $\varepsilon < \varepsilon_0$ and for all $\hat{\boldsymbol{x}}$ in ∂U_i, $\hat{\boldsymbol{x}} + \varepsilon\boldsymbol{X}_i(\hat{\boldsymbol{x}}) \in U_i$ and $\hat{\boldsymbol{x}} - \varepsilon\boldsymbol{X}_i(\hat{\boldsymbol{x}}) \notin U_i$. We set, for all positive δ', $U_i^{\delta'} = \{\boldsymbol{x} \in U_i,\ \mathrm{dist}(\boldsymbol{x},\partial U_i) < \delta'\}$, and $V_i^{\delta'} = \{\hat{\boldsymbol{x}} + s\boldsymbol{X}_i(\hat{\boldsymbol{x}}), \hat{\boldsymbol{x}} \in \partial U_i, 0 < s < \delta'\}$. We assume there exist $\hat{R}_i > 0$, $\theta_{\boldsymbol{X}}$, $0 < \theta_{\boldsymbol{X}} \leq \pi/2$, and δ_{0i}, $0 < \delta_{0i} \leq \hat{R}_i \sin\theta_{\boldsymbol{X}}$ such that $V_i^{\hat{R}} \subset L_i \cap U_i$ and $U_i^{\delta'} \subset V^{\delta'/\sin\theta_{\boldsymbol{X}}}$ for all positive $\delta' \leq \delta_{0i}$. Set $\tilde{R}_i := 1/\|\mathrm{div}\boldsymbol{X}_i\|_\infty$. We suppose $\delta_{0i} > \delta_i^l$.

We finally define, for all i, the linear form on $H_0^1(\Omega)$ by

$$\ell_i(u) := \frac{1}{\pi r_i^2}\int_{B(\boldsymbol{x}_i,r_i)} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \frac{1}{\pi}\int_{B(\boldsymbol{0},1)} u(\boldsymbol{x}_i + r_i\boldsymbol{y})\mathrm{d}\boldsymbol{y}.$$

We can now state our main theorem, namely the existence of a stable decomposition of $H_0^1(\Omega)$ whose upper bound is independent of $\frac{\max_i(H_i)}{\min_i(H_i)}$. This theorem therefore leads to a substantially sharper condition number estimate in the important case of an only locally shape regular decomposition, and is a major improvement of [3, Theorem 5.12], which only considered shape regular decompositions, albeit at the continuous level, in contrast to [2].

Theorem 1. *For u in $H_0^1(\Omega)$, there exists a stable decomposition $(u_i)_{0\le i\le N}$ of u, i.e., $u=\sum_{i=0}^N u_i$, u_0 in $P_1(\mathscr{T})\cap H_0^1(\Omega)$ and $u_i\in H_0^1(\Omega_i)$ such that*

$$\sum_{i=0}^{N}\|\nabla u_i\|^2_{L^2(\Omega_i)}\le C\|\nabla u\|^2_{L^2(\Omega)},$$

where $C=2C_1+2(1+C_1)C_2$ and

$$\begin{aligned}
C_1 &= \frac{1}{\tan\theta_{min}}\frac{\left(1+2\max_i(\frac{r_i}{H_{h,i}})\right)K(\frac{25}{6\pi}\max_i(\frac{d_i}{r_i})+2\pi\right)}{1-\left((2K+1)+(4K+1)\max_i(\frac{r_i}{H_{h,i}})\right)\max_i(\frac{r_i}{H_{h,i}})},\\
C_2 &= 2+8\lambda_2^2(N_c-1)^2(1+\max_i\frac{\hat{R}_i}{\tilde{R}_i})\max_i\frac{\delta_i^l}{\delta_i^s}\max_i\frac{\hat{R}_i}{\delta_i^s\sin\theta_{\boldsymbol{X}}}\\
&\quad+\frac{8}{3}\lambda_2^2(N_c-1)^2(1+\max_i\frac{\hat{R}_i}{\tilde{R}_i})\max_i\frac{\delta_i^l}{\delta_i^s}\max_i\frac{r_i^2}{\delta_i^s\hat{R}_i\sin\theta_{\boldsymbol{X}}}\times\\
&\quad\quad\times\max_i\left(\left(\left(\frac{H_i^2}{r_i^2}+\frac{1}{2}\right)^{\frac{1}{4}}+\frac{H_i}{\sqrt[4]{2}r_i}\right)^4-\frac{1}{2}-\frac{H_i^2}{r_i^2}-\frac{H_i^4}{2r_i^4}\right),
\end{aligned}$$

with λ_2 a universal constant depending only on the dimension, and being smaller than 6 in the two dimensional case we consider here.

Note that the condition $r_i\le\frac{H_{h,i}}{4K+1}$ implies that the denominator of C_1 is positive. The value of C_2 is also always positive.

3 Proof of Theorem 1

The proof is based on the continuous analysis in [3], but two results must be adapted to the situation of only locally shape regular decompositions: we first show in §3.1 how to construct the fine component, which is a technical extension of the result [3, Theorem 4.6] for the case where subdomain sizes H_i and overlaps δ_i can strongly depend on the subdomain index i. Second, we explain in §3.2 the construction of the coarse component in the case of strongly varying H_i and δ_i between subdomains, which is a non-trivial generalization of [3, Lemma 5.7]. With these two new results, and the remaining estimates from [3], the proof can be completed.

3.1 Constructing the fine component

We begin by establishing a stable decomposition when there is no coarse mesh.

Lemma 1. *Let u be in $H^1_0(\Omega)$. Then, there exist $(u_i)_{1\le i\le N}$, u_i in $H^1_0(\Omega_i)$ such that $u=\sum_{i=1}^N u_i$, and*

$$\sum_{i=1}^N \|\nabla u_i\|^2_{L^2(\Omega)} \le 2\|\nabla u\|^2_{L^2(\Omega)} + 8\lambda_2^2(N_c-1)^2\left(\sum_{i=1}^N (1+\frac{\hat{R}_i}{\tilde{R}_i})\frac{\delta_i^l}{\delta_i^s}\frac{\hat{R}_i}{\delta_i^s\sin\theta_{\boldsymbol{X}}}\|\nabla u\|^2_{L^2(U_i)}\right)$$
$$+8\lambda_2^2(N_c-1)^2\left(\sum_{i=1}^N (1+\frac{\hat{R}_i}{\tilde{R}_i})\frac{\delta_i^l}{\delta_i^s}\frac{1}{\delta_i^s\hat{R}_i\sin\theta_{\boldsymbol{X}}}\|u\|^2_{L^2(U_i)}\right), \tag{1}$$

where λ_2 is the universal constant of Theorem 1. We further have, for all $\eta>0$,

$$\sum_{i=1}^N \|\nabla u_i\|^2_{L^2(\Omega)} \le 2\|\nabla u\|^2_{L^2(\Omega)} + 8\lambda_2^2(N_c-1)^2\sum_{i=1}^N (1+\frac{\hat{R}_i}{\tilde{R}_i})\frac{\delta_i^l}{\delta_i^s}\frac{\hat{R}_i}{\delta_i^s\sin\theta_{\boldsymbol{X}}}\|\nabla u\|^2_{L^2(U_i)}$$
$$+\frac{8(1+\eta)}{3}\lambda_2^2(N_c-1)^2\sum_{i=1}^N (1+\frac{\hat{R}_i}{\tilde{R}_i})\frac{\delta_i^l}{\delta_i^s}\frac{r_i^2}{\delta_i^s\hat{R}_i\sin\theta_{\boldsymbol{X}}}\times$$
$$\times\left(\left(\left(\frac{H_i^2}{r_i^2}+\frac{1}{2}\right)^{\frac{1}{4}}+\frac{H_i}{\sqrt[4]{2}r_i}\right)^4-\frac{1}{2}-\frac{H_i^2}{r_i^2}-\frac{H_i^4}{2r_i^4}\right)\|\nabla u\|^2_{L^2(U_i)}$$
$$+8(1+\frac{1}{\eta})\pi\lambda_2^2(N_c-1)^2\sum_{i=1}^N (1+\frac{\hat{R}_i}{\tilde{R}_i})\frac{\delta_i^l}{\delta_i^s}\frac{H_i^2}{\delta_i^s\hat{R}_i\sin\theta_{\boldsymbol{X}}}|\ell_i(u)|^2. \tag{2}$$

Proof. We follow the proof of [3, Th. 4.6]. Let ρ be a $\mathscr{C}^\infty$ non-negative function whose support is included in the closed unit ball of $\mathbb{R}^2$ and whose L^1 norm is 1. Let $\rho_\varepsilon(\boldsymbol{x})=\rho(\boldsymbol{x}/\varepsilon)/\varepsilon^2$ for all $\varepsilon>0$. Let h_i be the characteristic function of the set $\{\boldsymbol{x}\in\mathbb{R}^2,\mathrm{dist}(\boldsymbol{x},U_i)<\delta_i/2\}$. Let $\phi_i=\rho_{\delta_i/2}*h_i$. The function ϕ_i is equal to 1 inside U_i, vanishes outside of $\{\boldsymbol{x}\in\mathbb{R}^2,\mathrm{dist}(\boldsymbol{x},U_i)<\delta_i\}$, and $\|\nabla\phi_i\|_{L^\infty(\mathbb{R}^2)}\le 2\|\nabla\rho\|_{L^1(\mathbb{R}^2;(\mathbb{R}^2,\|\cdot\|_2))}/\delta_i$. Here, $\|\nabla\rho\|_{L^1(\mathbb{R}^2;(\mathbb{R}^2,\|\cdot\|_2))}$ means $\int_{\mathbb{R}^2}\sqrt{\sum_{i=1}^2|\partial_i\rho|^2}\mathrm{d}\boldsymbol{x}$.

For i in $\mathbb{N}\cap[1,N]$, let $\psi_i=\phi_i\prod_{k=1}^{i-1}(1-\phi_k)$. We have $0\le\psi_i\le 1$, ψ_i zero in $\Omega\setminus\Omega_i$ and $\sum_i\psi_i=1$ in Ω. Set $u_i=\psi_i u$. The function u_i is in $H^1_0(\Omega_i)$ and $u=\sum_i u_i$. Following the proof of [3, Lemma 4.3], we get $\sum_{i=1}^N\|\nabla\psi_i(\boldsymbol{x})\|_2^2\le 2(N_C-1)\sum_{i=1}^N\|\nabla\phi_i(\boldsymbol{x})\|_2^2$. Therefore, for all $\boldsymbol{x}$ in Ω,

$$\sum_{i=1}^N\|\nabla\psi_i(\boldsymbol{x})\|_2^2\le 8(N_c-1)\|\nabla\rho\|^2_{L^1(\mathbb{R}^2;(\mathbb{R}^2,\|\cdot\|_2))}\sum_{i=1}^N\frac{\mathbb{1}_{\Omega_i\setminus U_i}(\boldsymbol{x})}{\delta_i^2},$$

where $\mathbb{1}_{\mathscr{O}}$ is the indicator function for the set $\mathscr{O}$. Since $\sum_i\|\nabla u_i\|^2_{L^2(\Omega)}\le 2\|\nabla u\|^2_{L^2(\Omega)}+2\int_\Omega|u(\boldsymbol{x})|^2\sum_i|\nabla\psi_i(\boldsymbol{x})|^2\mathrm{d}\boldsymbol{x}$, we get

$$\sum_{i=1}^N\|\nabla u_i\|^2_{L^2(\Omega)}\le 2\|\nabla u\|^2_{L^2(\Omega)}+4\lambda_2^2(N_c-1)^2\sum_{i=1}^N\int_{U_i}\mathbb{1}_{\{\mathrm{dist}(\boldsymbol{x},\partial U_i)<\delta_i^l\}}\frac{|u(\boldsymbol{x})|^2}{(\delta_i^s)^2}\mathrm{d}\boldsymbol{x},$$

with $\lambda_2 := 2\|\nabla\rho\|_{L^1(\mathbb{R}^2;(\mathbb{R}^2,\|\cdot\|_2))}$. Using the $W^{1,1}(\mathbb{R}^2)$ function $\rho(\boldsymbol{x}) = 1-\|\boldsymbol{x}\|_2$, we obtain the estimate $\lambda_2 = 6$. To get (1), we apply Lemma 4.5 in [3] to each U_i, and to obtain (2), we apply Lemma 5.10 from the same reference. $\square$

To obtain a stable decomposition with a coarse component, we want to construct u_0 in $P_1(\mathscr{T})$ such that for all i, $\ell_i(u_0) = \ell_i(u)$.

3.2 Constructing the coarse component

To construct u_0, we follow the ideas of [3, §5.2]. First, we define a special norm.

Definition 1. *Let $\mathscr{T}$ be the coarse mesh of the domain Ω. Let $\mathscr{B}'$ be the set of indices of the nodes of $\mathscr{T}$ located on the boundary[4] $\partial\Omega$. Let $\mathscr{B}$ be the set of the indices of the nodes that are neighbors to the nodes with index in $\mathscr{B}'$. Let $\mathscr{V}$ be the set of pairs of indices of neighboring nodes in $\mathscr{T}$ which are not on $\partial\Omega$. We define*

$$\begin{aligned} \|\cdot\|_{\mathscr{V},\mathscr{B}} : \mathbb{R}^N &\to \mathbb{R}^+, \\ \boldsymbol{y} &\mapsto \sqrt{\sum_{(i,j)\in\mathscr{V}} |y_i - y_j|^2 + \sum_{i\in\mathscr{B}} |y_i|^2}. \end{aligned}$$

When u is in $P_1(\mathscr{T}) \cap H_0^1(\Omega)$, set $\|u\|_{\mathscr{V},\mathscr{B}} := \|(u(\boldsymbol{x}_i))_{1\le i\le N}\|_{\mathscr{V},\mathscr{B}}$, where the $\boldsymbol{x}_i$ are the interior nodes of the mesh $\mathscr{T}$.

Lemma 2. *For u in $H_0^1(\Omega)$, there exists u_0 in $P_1(\mathscr{T}) \cap H_0^1(\Omega)$ such that, for all i in $\{1,\dots,N\}$, $\ell_i(u_0) = \ell_i(u)$ and*

$$\|\nabla u_0\|^2_{L^2(\Omega)} \le \frac{1}{\tan\theta_{min}} \frac{\left(1+2\max_i(\frac{r_i}{H_{h,i}})\right) K\left(\frac{25}{6\pi}\max_i(\frac{d_i}{r_i}) + 2\pi\right)}{1-\left((2K+1)+(4K+1)\max_i(\frac{r_i}{H_{h,i}})\right)\max_i(\frac{r_i}{H_{h,i}})}.$$

Proof. The results of [3, Lemmas 5.6,and 5.8] stand without modifications. Therefore u_0 exists, and we have

$$\|\nabla u_0\|^2_{L^2(\Omega)} \le \frac{1}{\tan\theta_{min}} \frac{1+2\max_i(\frac{r_i}{H_{h,i}})}{1-\left((2K+1)+(4K+1)\max_i(\frac{r_i}{H_{h,i}})\right)\max_i(\frac{r_i}{H_{h,i}})} \|u\|^2_{\mathscr{V},\mathscr{B}}.$$

Note that the condition $r_i \le \frac{H_{h,i}}{4K+1}$ implies the second denominator in the above equation positive.

It remains to compare $\|u\|^2_{\mathscr{V},\mathscr{B}}$ and $\|\nabla u\|^2_{L^2(\Omega)}$. We need to adapt the proof of [3, Lemma 5.7]. We can suppose without any loss of generality that u is in $\mathscr{C}^\infty(\overline{\Omega})$. Let i,j in $\{1,\dots,N\}$ be indices of neighboring nodes of $\mathscr{T}$. Let $\boldsymbol{d}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j$, and $d_{ij} = \|\boldsymbol{d}_{ij}\|$. We have for all $(i,j) \in \mathscr{V}$

[4] Because of the homogenous Dirichlet condition on the boundary $\partial\Omega$, the nodes whose indices are in $\mathscr{B}'$ are not associated to a degree of freedom, therefore $\mathscr{B}'$ and $\{1,\dots,N\}$ have empty intersection.

$$
\begin{aligned}
|\ell_i(u)-\ell_j(u)|^2 &= \frac{1}{\pi^2}\left(\int_{B(\mathbf{0},1)} (u(\boldsymbol{x}_i+r_i\boldsymbol{y})-u(\boldsymbol{x}_j+r_j\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right)^2 \\
&\leq \frac{1}{\pi}\int_{B(\mathbf{0},1)}\int_0^1 \|\nabla u\big(t(\boldsymbol{x}_i+r_i\boldsymbol{y})+(1-t)(\boldsymbol{x}_j+r_j\boldsymbol{y})\big)\|_2^2 \|\boldsymbol{x}_i-\boldsymbol{x}_j+(r_i-r_j)\boldsymbol{y}\|_2^2 \mathrm{d}t\mathrm{d}\boldsymbol{y} \\
&\leq \frac{(d_{ij}+|r_i-r_j|)^2}{\pi}\int_{B(\mathbf{0},1)}\int_0^1 \|\nabla u\big(t(\boldsymbol{x}_i+r_i\boldsymbol{y})+(1-t)(\boldsymbol{x}_j+r_j\boldsymbol{y})\big)\|_2^2 \mathrm{d}t\mathrm{d}\boldsymbol{y} \\
&\leq \frac{(d_{ij}+|r_i-r_j|)^2}{\pi}\int_{T_{i,j}} \|\nabla u(\boldsymbol{y}')\|_2^2 \int_0^1 \frac{\mathbb{1}_{\{\|\boldsymbol{y}'-t\boldsymbol{x}_i-(1-t)\boldsymbol{x}_j\|\leq tr_i+(1-t)r_j\}}}{(tr_i+(1-t)r_j)^2}\mathrm{d}t\mathrm{d}\boldsymbol{y}',
\end{aligned}
$$

where the tube $T_{i,j}$ is the convex hull of $B(\boldsymbol{x}_i,r_i)\cup B(\boldsymbol{x}_j,r_j)$. We get

$$
\begin{aligned}
&\max_{\boldsymbol{y}'\in\mathbb{R}^2}\int_0^1 \frac{\mathbb{1}_{\{\|\boldsymbol{y}'-t\boldsymbol{x}_i-(1-t)\boldsymbol{x}_j\|\leq tr_i+(1-t)r_j\}}}{(tr_i+(1-t)r_j)^2}\mathrm{d}t \\
&\quad= \max_{(s,s')\in\mathbb{R}^2}\int_0^1 \frac{\mathbb{1}_{\{\sqrt{(s-td_{ij})^2+s'^2}\leq tr_i+(1-t)r_j\}}}{(tr_i+(1-t)r_j)^2}\mathrm{d}t \\
&\quad= \max_{s\in[-r_j,d_{ij}+r_i]}\int_0^1 \frac{\mathbb{1}_{\{|s-td_{ij}|\leq tr_i+(1-t)r_j\}}}{(tr_i+(1-t)r_j)^2}\mathrm{d}t \\
&\quad\leq \max_{s\in[-r_j,d_{ij}+r_i]}\int_{\frac{s-r_j}{d_{ij}+(r_i-r_j)}}^{\frac{s+r_j}{d_{ij}-(r_i-r_j)}} \frac{1}{(tr_i+(1-t)r_j)^2}\mathrm{d}t \\
&\quad= \max_{s\in[-r_j,d_{ij}+r_i]} -\frac{1}{r_i-r_j}\left[\frac{1}{(tr_i+(1-t)r_j)}\right]_{\frac{s-r_j}{d_{ij}+(r_i-r_j)}}^{\frac{s+r_j}{d_{ij}-(r_i-r_j)}} \\
&\quad= \max_{s\in[-r_j,d_{ij}+r_i]}\left(\frac{2}{d_{ij}r_j+s(r_i-r_j)}\right) \\
&\quad= \frac{2}{\min(r_i,r_j)(d_{ij}-|r_i-r_j|)}.
\end{aligned}
$$

Since $d_{ij}\geq H_{h,i}\geq 4\max(r_i,r_j)$, we have

$$
|\ell_i(u)-\ell_j(u)|^2 \leq \frac{25d_{ij}}{6\pi\min(r_i,r_j)}\|\nabla u\|_{L^2(T_{ij})}^2. \tag{3}
$$

If i is in the boundary set of the coarse mesh, then the node $\boldsymbol{x}_i$ is neighbor to a node $\boldsymbol{x}_{i'}$ located on $\partial\Omega$. Note that i' lies outside of the range $\{1,\ldots,N\}$. Using [3, Eqs (5.7) and (5.9)], we get

$$
\sum_{i\in\mathscr{B}}|\ell_i(u)|^2 \leq \left(\sum_{i\in\mathscr{B}}\frac{4\|\boldsymbol{x}_i-\boldsymbol{x}_{i'}\|}{\pi r_i}\int_{T_i'}\|\nabla u(\boldsymbol{x})\|^2\mathrm{d}\boldsymbol{x}\right)+2K\pi\|\nabla u\|_{L^2(\Omega)}^2, \tag{4}
$$

where T_i' is the convex hull of $B(\boldsymbol{x}_i,r_i)\cup B(\boldsymbol{x}_{i'},r_i)$. We sum inequality (3) over all i,j in the neighbor set and combine the resulting inequality with equation (4). Since

$\max(r_i, r_j) \leq H'_{h,i}/2 \leq \min(H_{h,i}, H_{h,j}))/2$, no point can belong to more than K tubes $T_{i,j}$ or T'_i. Therefore, $\|u\|^2_{\mathscr{V},\mathscr{B}} \leq K\big(25\max_i(d_i/r_i)/(6\pi)+2\pi\big)\|\nabla u\|^2_{L^2(\Omega)}$. This concludes the proof. $\square$

To prove Theorem 1, we use Lemma 2 to construct the coarse component u_0. We then apply Lemma 1 to $u-u_0$ to get the fine components u_i. The terms in $\ell_i(u)$ vanish.

4 Conclusion

We have proved the existence of a stable decomposition of the Sobolev space $H^1_0(\Omega)$ in the presence of a coarse mesh when the domain decomposition is only guaranteed to be locally shape regular. We provided an explicit upper bound for the stable decomposition that depends neither on $\max_i(H_i)/\min_i(H_i)$, nor on the number of subdomains. This would not have been possible without the explicit upper bounds provided in [3]. This shows that deriving such explicit upper bounds can be important for problems arising naturally in applications, *e.g.*, load balanced domain decompositions with local refinement.

References

[1] Tony F. Chan and Tarek P. Mathew. Domain decomposition algorithms. In *Acta Numerica 1994*, pages 61–143. Cambridge University Press, 1994.

[2] Maksymilian Dryja and Olof B. Widlund. An additive variant of the Schwarz alternating method for the case of many subregions. Technical Report 339, also Ultracomputer Note 131, Department of Computer Science, Courant Institute, 1987.

[3] Martin J. Gander, Laurence Halpern, and Kévin Santugini-Repiquet. Continuous Analysis of the Additive Schwarz Method: a Stable Decomposition in H^1. *Submitted*, 2011. URL `http://hal.archives-ouvertes.fr/hal-00462006/fr/`.

[4] Alfio Quarteroni and Alberto Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford Science Publications, 1999.

[5] Hermann A. Schwarz. Über einen Grenzübergang durch alternierendes Verfahren. *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, 15: 272–286, May 1870.

[6] Barry F. Smith, Petter E. Bjørstad, and William Gropp. *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*. Cambridge University Press, 1996.

[7] Andrea Toselli and Olof Widlund. *Domain Decomposition Methods - Algorithms and Theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer, 2004.

[8] Jinchao Xu. Iterative methods by space decomposition and subspace correction. *SIAM Review*, 34(4):581–613, December 1992.